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AVERAGED CHARACTERISTICS OF A NONLINEAR COMPOSITE

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The question on the calculation of certain effective characteristics of a nonlinear composite expressed in terms of integrals of the solution of a cellular problem is considered. The problem for the case of weak nonlinearities or weak fields has been investigated. Calculations of the effective characteristics of the composite have been made.

Introduction. In the electrical engineering industry, materials with a nonlinear dependence of the dielectric constant on the electric field applied are used. To create materials with optimal characteristics, composites are used, and in this connection the question on the calculation of their nonlinear characteristics arises. The specific feature of the problem is the fact that the quantities expressed in terms of the effective characteristics of the composite should be optimal. Of interest is the solution of the problem for the case of weak nonlinearities or weak fields.

In the present work, much consideration is given to the theoretical justification of the fact that the controllability coefficient of a nonlinear material decreases slightly upon the introduction into it of particles of a linear (having zero controllability) material. The investigation is carried out with the example of a two-dimensional problem. Physically, this corresponds to a laminated material or a material filled with highly elongated particles (fiber).

Controllability Coefficient of the Composite. Of practical interest is the so-called relative controllability of a nonlinear dielectric

$$K = \frac{\varepsilon(0) - \varepsilon(E)}{\varepsilon(0)} . \tag{1}$$

This quantity characterizes the changeability of the dielectric properties of a material when applied to a constant stress and makes it possible to use it in electrotechnical control devices. It practice, the function $\varepsilon(E)$ is usually taken in the form

$$\varepsilon(E) = \varepsilon_0 + \mu E^2 , \tag{2}$$

although the plot of the function $\varepsilon(E) = \dfrac{\varepsilon_0}{1 + bE^2}$ better corresponds to experimental data.

For most materials, $\mu < 0$ [1]. In choosing $\varepsilon(E)$ in the form (2), formula (1) is written as $K = \dfrac{\mu}{\varepsilon_0}E^2$, and (2) can be given in the form $\varepsilon(E) = \varepsilon_0(1 + KE^2)$. The quantity $-\mu/\varepsilon_0 > 0$ if $\mu < 0$. The material properties of a composite (inhomogeneous, see Fig. 1) material depend on the spatial variable $\mathbf{x} = (x, y)$. Consequently, in (2) ε_0 and μ are functions of $\mathbf{x}$ and for the composite relation (2) is written in the form

$$\varepsilon(\mathbf{x}, \nabla\varphi) = \varepsilon_0(\mathbf{x}) + \mu(\mathbf{x}) \left|\nabla\varphi\right|^2 . \tag{3}$$

The induction vector $\mathbf{I}$ is related to the function $\nabla\varphi$ by the formula $\mathbf{I} = \varepsilon(\mathbf{x}, \nabla\varphi)\nabla\varphi$ and satisfies the equation

$$\text{div}\,\mathbf{I} = 0 , \tag{4}$$

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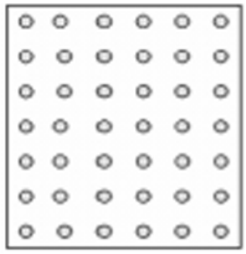

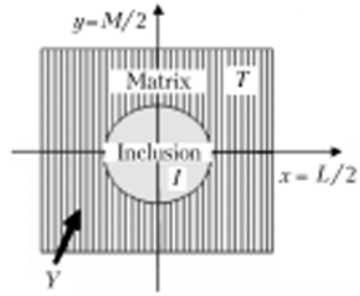

Fig. 1. Model of a two-dimensional composite material (on the left) and its periodicity cell Y (on the right).

which can be rewritten in the form

$$\mathrm{div}\,[\varepsilon\,(\mathbf{x}, \nabla\varphi)\,\nabla\varphi] = 0\,. \qquad (5)$$

Having solved problem (5) with proper boundary conditions, one can determine the potential $\varphi(\mathbf{x})$. However, for an inhomogeneous medium this problem is practically unsolvable. Discretization of the region in Fig. 1 by a mesh (finite-element or computational) will lead to a problem of very large dimension. As an example, we will show that for the calculation of a composite of the considered type performed in [2] on large-scale computers a mesh selecting 3–5 finite elements per inclusion was used. Such a number of finite elements is evidently insufficient for calculating the material characteristics of the composite.

Composite materials formed from small components are considered in practice as homogeneous material with their own material characteristics (called global, macroscopic, effective, or averaged characteristics). They differ from the characteristics of the composite components and at the same time are uniquely determined by them.

Characteristic Values of the Characteristics of Composite Components. The matrix material is a nonlinear ferroelectric with the following characteristics determined experimentally in [1] (in dimensionless variables the dielectric constant of vacuum is equal to 1): ε_0 = 2000–3000, μ = 400–600. The material of inclusions is a linear dielectric with ε_0 = 3–5 and $\mu = 0$. Materials with such characteristics of their components should be regarded as high-contrast ones.

Averaging in the Small-Nonlinearity Problem. Many authors have made attempts to calculate averaged characteristics of nonlinear composites of the in type described [1, 3, 4]. In the present work, the problem of their determination is solved for a composite with a periodic structure (see Fig. 1) in the case of a small nonlinearity or a weak field (weak in the sense that the quantity $K \ll 1$ or $E \ll 1$).

Let us consider Eq. (5) with the coefficient ε in the form

$$\varepsilon\left(\mathbf{x}, \,|\nabla\varphi|\,\right) = \varepsilon_0\,(\mathbf{x}) + \lambda\mu\,(\mathbf{x})\,|\nabla\varphi|^2\,. \qquad (6)$$

Such nonlinearity is called small, and it also arises when the field is small.

Consider a two-dimensional composite with a periodic structure whose periodicity cell represents a rectangle with center at the origin of coordinates and sides of length L and M. For simplicity, let us assume that in the periodicity cell one inclusion axisymmetric about the coordinate axes is contained (Fig. 1). Let stresses $\pm UM/2$ (these values correspond to the averaged stress U) be applied to the horizontal faces of the periodicity cell, and let periodicity conditions be set on the vertical faces (Fig. 1). From the symmetry and equivalence of the periodicity cells the boundary conditions

$$\varphi\,(x, \pm\, M/2) = \pm\, UM/2\,, \qquad (7)$$

$$\frac{\partial\varphi}{\partial\mathbf{n}}\,(\pm\, L/2, y) = 0 \qquad (8)$$

follow.

The total induction vector flux $\mathbf{I} = \varepsilon(|\nabla\varphi|)\nabla\varphi$ through the periodicity cell is equal to the sum of the fluxes through its sides. The flux through the upper face $\Gamma = \{-L/2 < x < L/2,\ y = M/2\}$ of the periodicity cell Y corresponding to the potential difference U in (7) is equal to

$$D_{\text{total}} = \int_{\Gamma} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)\frac{\partial\varphi}{\partial\mathbf{n}}\, d\mathbf{x}\,. \tag{9}$$

Multiplying (5) by φ and integrating by parts, we obtain, in view of (7) and (8),

$$-\int_{Y} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)|\nabla\varphi|^2\, d\mathbf{x} + \int_{x=\pm L/2} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)\varphi\, d\mathbf{x} + \int_{y=\pm M/2} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)\varphi\, d\mathbf{x} = 0\,, \tag{10}$$

hence

$$-\int_{Y} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)|\nabla\varphi|^2\, d\mathbf{x} + \int_{y=\pm L/2} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)\frac{\partial\varphi}{\partial\mathbf{n}}[\pm\, UL/2]\, d\mathbf{x} = 0\,, \tag{11}$$

where [] denotes a "jump." The fluxes through the upper and lower faces Y are equal, by virtue of which we obtain from (10) and (11)

$$\int_{y=\pm M/2} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)\nabla\varphi d\mathbf{x} = UL \int_{y=M/2} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)\frac{\partial\varphi}{\partial\mathbf{n}}\, d\mathbf{x}\,. \tag{12}$$

From (9), (11), and (12) (provided that the flux corresponds to the potential difference U) we have

$$D_{\text{total}} = \int_{\Gamma} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)\frac{\partial\varphi}{\partial\mathbf{n}}\, d\mathbf{x} = \frac{1}{UL}\int_{Y} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)|\nabla\varphi|^2\, d\mathbf{x}\,.$$

The specific flux F (flux per unit length of the horizontal side of the periodicity cell) equals

$$F = \frac{D_{\text{total}}}{M} = \frac{1}{ULM}\int_{Y} \varepsilon\left(\mathbf{x},\ |\nabla\varphi|\right)|\nabla\varphi|^2\, d\mathbf{x}\,. \tag{13}$$

The formulas obtained are general. Consider the case of weak nonlinearity (6). Following [5], we seek a solution to the problem (5), (7), (8) in the form

$$\varphi = \varphi_0 + \lambda\varphi_1 + \dots\ . \tag{14}$$

Substituting (14) into (5), (7), (8) and equating the terms at λ^0 and λ, we obtain equations for the zeroth-order term (term at λ^0)

$$\operatorname{div}\left[\varepsilon_0(\mathbf{x})\,\nabla\varphi_0\right] = 0\,,\quad \varphi_0(x, \pm\, M/2) = \pm\, U/2\,,\quad \frac{\partial\varphi_0}{\partial\mathbf{n}}(\pm\, L/2, y) = 0 \tag{15}$$

and the first-order term (corresponds to λ)

$$\operatorname{div}\left[\varepsilon_0(\mathbf{x})\,\nabla\varphi_1 + \mu(\mathbf{x})\,|\nabla\varphi_0|^2\,\nabla\varphi_0\right] = 0\,,\quad \varphi_1(x, \pm\, M/2) = 0\,,\quad \frac{\partial\varphi_0}{\partial\mathbf{n}}(\pm\, L/2, y) = 0\,. \tag{16}$$

Substituting (14) into (13) and holding constant the λ^0- and λ-order terms leads to the equation

$$F = \frac{1}{ULM} \int_Y \varepsilon_0 (\mathbf{x}) \; |\nabla\varphi_0|^2 \, d\mathbf{x} + \frac{\lambda}{ULM} \int_Y \Big[\varepsilon_0 (\mathbf{x}) \, \nabla\varphi_0 \nabla\varphi_1 + \mu (\mathbf{x}) \; |\nabla\varphi_0|^4 \Big] d\mathbf{x} \,. \tag{17}$$

Multiplying the first equation from (15) by φ_1 and integrating by parts, we get

$$- \int_Y \varepsilon_0 (\mathbf{x}) \, \nabla\varphi_0 \nabla\varphi_1 d\mathbf{x} + \int_{\partial Y} \varepsilon_0 (\mathbf{x}) \, \frac{\partial\varphi_0}{\partial\mathbf{n}} \, \varphi_1 d\mathbf{x} = 0 \,. \tag{18}$$

The integral $\int_{y=\pm M/2} \varepsilon_0 (\mathbf{x}) \, \frac{\partial\varphi_0}{\partial\mathbf{n}} \varphi_1(x, \, y) d\mathbf{x} = 0$ by virtue of the condition $\varphi_1(x, \, \pm M/2) = 0$ from (16), and the integral $\int_{x=\pm L/2} \varepsilon_0 (\mathbf{x}) \, \frac{\partial\varphi_0}{\partial\mathbf{n}}(x, \, y)\varphi_1(x, \, y) d\mathbf{x} = 0$ by virtue of the condition $\frac{\partial\varphi_0}{\partial\mathbf{n}}(\pm L/2, \, y) = 0$ from (15). Then the boundary integral in (18) is equal to zero, from which it follows that $-\int_Y \varepsilon_0(\mathbf{x})\nabla\varphi_0\nabla\varphi_1 d\mathbf{x} = 0$. Making use of this equality, we can rewrite (17) in the form

$$F = \frac{1}{UML} \int_Y \varepsilon_0 (\mathbf{x}) \; |\nabla\varphi_0|^2 \, d\mathbf{x} + \frac{\lambda}{UML} \int_Y \mu (\mathbf{x}) \; |\nabla\varphi_0|^4 \, d\mathbf{x} \,. \tag{19}$$

By virtue of (19) the specific flux of the electric field through the periodicity cell Y has the form $F = F_0 + \lambda F_1$, where

$$F_0 = \frac{1}{UML} \int_Y \varepsilon_0 (\mathbf{x}) \; |\nabla\varphi_0|^2 \, d\mathbf{x} \,, \quad F_1 = \frac{1}{UML} \int_Y \mu (\mathbf{x}) \; |\nabla\varphi_0|^4 \, d\mathbf{x} \,.$$

The averaged dielectric constant D of the composite can be introduced as a ratio of the total flux F of the induction vector to the corresponding potential difference U. In the case under consideration,

$$D = \frac{F}{U} = D_0 + \lambda D_1 \,, \tag{20}$$

where $D_0 = F_0/U$, $D_1 = F_1/U$. Let us introduce the function $N(x, \, y)$ as a solution to the cell problem (15) at $U = 1$. Then $\varphi_0 = UN$ and

$$D_0 = \frac{1}{ML} \int_Y \varepsilon_0 (\mathbf{x}) \; |\nabla N|^2 \, d\mathbf{x} \,, \quad D_1 = \frac{1}{ML} \int_Y \mu (\mathbf{x}) \, U^2 \; |\nabla N|^4 \, d\mathbf{x} \,. \tag{21}$$

Formula (20) can be written in the form

$$D = A + BU^2 \,, \tag{22}$$

where $A = D_0$, $B = \frac{1}{ML} \int_Y \mu(\mathbf{x}) \, |\nabla N|^4 d\mathbf{x}$.

Consider a composite of the form "nonlinear-material matrix — linear-material inclusion." In this case, $\mu(\mathbf{x}) \neq 0$ in the matrix and $\mu(\mathbf{x}) = 0$ in the inclusion, and the second formula from (21) takes the form

$$D_1 = \frac{1}{ML} \int_T \mu U^2 \; |\nabla N|^4 \, d\mathbf{x} \,, \tag{23}$$

i.e., the integral is taken over the matrix. For the controllability coefficient of the composite K_{mix}, we obtain the formula

$$K_{\text{mix}} = \frac{A}{B} U^2 = \lambda U^2 \frac{\int_T \mu \, |\nabla N|^4 \, d\mathbf{x}}{\int_T \varepsilon_0 \, |\nabla N|^2 \, d\mathbf{x}} . \tag{24}$$

Weak fields. Let at the sample boundary $\varphi(x, \pm L/2) = \pm \xi UL/2$, where $\xi << 1$. Then inside the composite the field has the same low order: $\varphi = \xi\varphi_1$, and expression (6) takes the form

$$\varepsilon\,(\mathbf{x}, \,|\nabla\varphi|) = \varepsilon_0\,(\mathbf{x}) + \mu\,(\mathbf{x})\,\xi^2 \,|\nabla\varphi_1| \, , \tag{25}$$

and (5) takes the form (can be divided by ξ)

$$\operatorname{div}\left[\left(\varepsilon_0\,(\mathbf{x}) + \mu\,(\mathbf{x})\,\xi^2 \,|\nabla\varphi_1|^2\right)\nabla\varphi_1\right] = 0 \, . \tag{26}$$

Comparing (25), (26) to (5), (6), we see that they coincide if we assume $\lambda = \xi^2$. Thus, formula (23) for the nonlinear part of the averaged dielectric constant is valid for weak fields without the smallness condition $\mu(\mathbf{x})$.

The presence of the term "the fourth degree of the field intensity" in (19) agrees with the results of [3, 4], where the consideration was carried out on the physical level of rigor. The above analysis of the problem has been performed with the use of the rigorous method of averaging [6] and justifies formula (20) only for weak nonlinearities or fields. There is no reason to extend these formulas to the general case.

Analysis of the Controllability Coefficient of a High-Contrast Composite. The averaged controllability coefficient (demonstrated by a composite considered as a certain homogeneous material) corresponding to (22) is equal to

$$K_{\text{mix}} = \frac{B}{A} E^2 . \tag{27}$$

From (15) and (21) it is seen that the main difficulty in calculating the characteristics is the calculation of the function $N(x, y)$. Let us investigate this issue as applied to high-contrast composites.

Consider the periodicity cell of a composite (matrix T with inclusion I, Fig. 1) The dielectric constant of the inclusion is small compared to such for the matrix. We assume: $\varepsilon(\mathbf{x}) \approx 0$ in the inclusion I, $\varepsilon(\mathbf{x}) = \varepsilon_0$ in the matrix T. The inclusion material is linear: $\mu(\mathbf{x}) = 0$ in the inclusion, $\mu(\mathbf{x}) = \mu$ in the matrix T. For this case, from (21) and (27) we obtain the following formulas:

$$A = \varepsilon_0 I_2 \, , \quad B = \mu I_4 \, , \quad K_{\text{mix}} = \frac{\mu}{\varepsilon_0} J E^2 \, , \tag{28}$$

where

$$I_2 = \frac{1}{|Y|} \int_T |\nabla N\,(\mathbf{x})|^2 \, d\mathbf{x} \, ; \quad I_4 = \frac{1}{|Y|} \int_T |\nabla N\,(\mathbf{x})|^4 \, d\mathbf{x} \, ; \quad J = I_4 / I_2 \, .$$

Integration is carried out only with respect to the matrix, since $\varepsilon(\mathbf{x}) \approx 0$ and $\mu(\mathbf{x}) = 0$ in the inclusion I.

The controllability coefficient of a pure ferroelectric is equal to $\frac{\mu}{\varepsilon_0} E^2$, by virtue of which J from (28) corresponds to the ratio of the controllability coefficient of the composite K_{mix} to the controllability coefficient of the pure ferroelectric

$$J = \frac{K_{\text{mix}}}{K}. \qquad (29)$$

The right-hand side of formula (28) for K_{mix} consists of the cofactor μ/ε_0 depending on the material constants of the matrix and the cofactor $J = \dfrac{\int_T |\nabla N|^4 d\mathbf{x}}{\int_T |\nabla N|^2 d\mathbf{x}}$, depending only on the inclusion geometry. To estimate the possible values of the controllability coefficient, one has to investigate the quantity J provided that $N(x)$ is the solution of problem (15).

Estimates for the Averaged Controllability Coefficient of a High-Contrast Composite. The controllability coefficient of a high-contrast composite, according to formulas (28) and (29), is expressed in terms of two integral functionals I_2 and I_4. In a number of works [3], it is assumed that $I_4 \approx I_2^2$. Let us estimate the integrals I_2 and I_4 and see to what extent the above hypothesis is justified.

Lower estimate for an arbitrary concentration of components. Let us apply the Cauchy–Bunyakowsky inequality to the integral $\int_T |\nabla N(\mathbf{x})|^2 d\mathbf{x}$ in the form

$$\int_T |\nabla N(\mathbf{x})|^2 d\mathbf{x} \le \left(\int_T |\nabla N(\mathbf{x})|^4 d\mathbf{x}\right)^{1/2} \left(\int_T d\mathbf{x}\right)^{1/2} = \left(\int_T |\nabla N(\mathbf{x})|^4 d\mathbf{x}\right)^{1/2} |T|^{1/2}.$$

Squaring and dividing the result by $|Y|^2$, we get

$$\left(\frac{1}{|Y|}\int_Y |\nabla N(\mathbf{x})^2|^4 d\mathbf{x}\right)^2 \le \frac{1}{|Y|}\int_Y |\nabla N(\mathbf{x})|^4 d\mathbf{x} \frac{|T|}{|Y|},$$

form which it follows that $I_2^2 \le I_4 \frac{|T|}{|Y|}$ or $I_4 \le I_2^2 \frac{|Y|}{|T|}$. From this inequality it follows that the coefficient in formula (28) can be estimated below as

$$\frac{\mu}{\varepsilon_0} J = \frac{B}{A} \ge \frac{\mu}{\varepsilon_0} I_2 \frac{|Y|}{|T|}.$$

Note that $|T|/|Y| = 1 - q$, then

$$\frac{\mu}{\varepsilon_0} J = \frac{B}{A} \ge \frac{\mu}{\varepsilon_0} \frac{I_2}{1-q}.$$

Lower estimate for the controllability of a low-filled three-dimensional composite. It is known [7, 8] that for a low-filled three-dimensional composite

$$A = \varepsilon_0 (1 - 1.5q), \qquad (30)$$

hence $I_2 = 1 - 1.5q$. Consequently,

$$\frac{\mu}{\varepsilon_0} J = \frac{B}{A} \ge \frac{\mu}{\varepsilon_0} \frac{I_2}{1-q} = \frac{\mu}{\varepsilon_0} \frac{1-1.5q}{1-q} \approx \frac{\mu}{\varepsilon_0}(1 - 1.5q + q) = \frac{\mu}{\varepsilon_0}(1 - 0.5q). \qquad (31)$$

With increasing volume content of the dielectric this expression decreases more slowly than the averaged dielectric constant.

Upper estimate for the composite controllability. Let us apply the Cauchy–Bunyakowsky inequality to the integral $\int_T |\nabla N(\mathbf{x})|^4 d\mathbf{x}$ in the following form:

$$\int_T |\nabla N(\mathbf{x})|^4 d\mathbf{x} \le \left(\int_T |\nabla N(\mathbf{x})|^8 d\mathbf{x}\right)^{1/2} \left(\int_T d\mathbf{x}\right)^{1/2} = \left(\int_T |\nabla N(\mathbf{x})|^8 d\mathbf{x}\right)^{1/2} |T|^{1/2} .$$

Then

$$\left(\int_T |\nabla N(\mathbf{x})|^4 d\mathbf{x}\right)^2 \le \int_T |\nabla N(\mathbf{x})|^8 d\mathbf{x} \; |T| \; .$$

We divide the last inequality by $|Y|^2$ and obtain

$$\left(\frac{1}{|Y|}\int_Y |\nabla N(\mathbf{x})|^4 d\mathbf{x}\right)^2 \le \frac{1}{|Y|}\int_Y |\nabla N(\mathbf{x})|^8 d\mathbf{x} \frac{|T|}{|Y|} ,$$

hence

$$I_4^2 \le \frac{1}{|Y|}\int_Y |\nabla N(\mathbf{x})|^8 d\mathbf{x} \frac{|T|}{|Y|} \quad \text{or} \quad I_4^2 \le \frac{1}{|Y|}\int_Y |\nabla N(\mathbf{x})|^8 d\mathbf{x}\,(1-q) .$$

One more upper estimate can be obtained by applying the Cauchy–Bunyakowsky to the integral $\int_T |\nabla N(\mathbf{x})|^4 d\mathbf{x}$ in the form

$$\int_T |\nabla N(\mathbf{x})|^4 d\mathbf{x} = \int_T |\nabla N(\mathbf{x})|^3 \; |\nabla N(\mathbf{x})| \; d\mathbf{x} \le \left(\int_T |\nabla N(\mathbf{x})|^6 d\mathbf{x}\right)^{1/2} \left(\int_T |\nabla N(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2} .$$

As a result of squaring and dividing by $|Y|^2$, we get

$$\left(\frac{1}{|Y|}\int_Y |\nabla N(\mathbf{x})|^4 d\mathbf{x}\right)^2 \le \left(\frac{1}{|Y|}\int_T |\nabla N(\mathbf{x})|^8 d\mathbf{x}\right)\left(\frac{1}{|Y|}\int_T |\nabla N(\mathbf{x})|^2 d\mathbf{x}\right).$$

The last cofactor on the right-hand side of this inequality represents the averaged dielectric constant A from (22) at "zero" intensity of the electric field, hence

$$I_4^2 \le A\left(\frac{1}{|Y|}\int_Y |\nabla N(\mathbf{x})|^6 d\mathbf{x}\right).$$

Numerical Analysis of the Problem. More detailed information on the behavior of the averaged controllability coefficient can be obtained from the numerical analysis of the problem.

The cell problem ((15) at $U = 1$) was solved for the case of a circular inclusion located in the center of a periodicity cell with dimensions $L = M = 1$. The solution $N(\mathbf{x})$ was sought by minimizing the functional

TABLE 1. Values of Integrals Depending on the Volume Content of Dielectric Inclusions q (data for a high-contrast composite)

q	$\sqrt{I_8}$	I_4	I_2	$\frac{I_4}{I_2^2}$	J
0.09	0.931169	0.860297	0.874860	1.124013	0.983354
0.19	0.857418	0.677497	0.707552	1.353288	0.957522
0.35	0.703303	0.481319	0.523116	1.758886	0.920101
0.46	0.604010	0.367643	0.407542	2.213504	0.902097

TABLE 2. Values of Integrals Depending on the Ratio $\varepsilon_{\text{diel}}/\varepsilon_0$, $q = 0.19$ (data for a high-contrast composite)

$\frac{\varepsilon_{\text{diel}}}{\varepsilon_0}$	$\frac{A}{\varepsilon_0}$	$\frac{A}{\varepsilon_0 I_2^2}$	J
0	0.707552	1.353288	0.957522
0.001	0.709813	1.344564	0.954389
0.5	0.890474	0.944252	0.840831
0.9	0.980478	0.834691	0.818396

$\int_Y \varepsilon_0(\mathbf{x})|\nabla N|^2 d\mathbf{x}$ from the set of functions satisfying the boundary conditions $N(x, \pm 1/2) = \pm 1/2$ and $\frac{\partial N}{\partial \mathbf{n}}(\pm 1/2, y) = 0$. Using the symmetry, we can pass to the problem on 1/4 of the periodicity cell.

The derivatives were approximated by the finite differences $(N^{i+1,j} - N^{i,j})/h$ and $(N^{i,j+1} - N^{i,j})/h$. As a result, there appeared a quadratic form from the finite number of variables which was minimized by the gradient descent method. Once an approximate solution of problem (15) was obtained, we calculated the approximate values of integrals (21).

The results of the calculations are presented in Tables 1 and 2. Table 1 gives the values of the integrals I_2, I_4, and $I_8 = \frac{1}{|Y|}\int_T |\nabla N|^8 d\mathbf{x}$ depending on the volume content of dielectric inclusions q at $\varepsilon_{\text{diel}} = 0$. The value of I_2 is equal to the ratio of the averaged dielectric constant of the composite to the dielectric constant of the pure dielectric (matrix material).

From Table 1 it is seen that I_2 and I_4 decrease by nearly one-half when the volume content of inclusions changes from 0.09 to 0.46. By virtue of the results from Table 1, it can be assumed that $I_4 \approx I_2$ rather than $I_4 \approx I_2^2$. The ratio I_4/I_2 (equal to the controllability coefficient of the composite upon multiplying by μ/ε_0) changed from 0.98 to 0.90 as the volume content of inclusions increased from 0.09 to 0.46; i.e., the controllability coefficient of the composite is the least varying quantity compared to the other quantities given in Table 1, in particular, the averaged dielectric constant. This conclusion is confirmed by formulas (30) and (31). The averaged dielectric constant of the low-filled composite A depending on the volume content of the dielectric q decreases as $1 - 1.5q$, and the lower estimate for the controllability coefficient of the composite decreases as $1 - 0.5q$.

Table 2 shows the dependence of the quantities A/ε_0 (ratio of the averaged dielectric constant of the composite to the dielectric constant of the pure ferroelectric — matrix material) and J on the $\varepsilon_{\text{diel}}/\varepsilon_0$ ratio.

Controllability of a Laminated Composite. Consider a problem on the determination of the dielectric characteristics of a laminated composite in the direction transverse to the layers. This problem is of both theoretical (since the solution can be constructed in explicit form) and practical interest (due to the relative simplicity of creating laminated composites with given characteristics [5]). The dependence of the dielectric constant on the field can be written in terms of the value of the electric field $E = |\nabla\varphi|$ or the value of the induction vector magnitude $|\mathbf{I}| = |\varepsilon(|\nabla\varphi|)\nabla\varphi|$. In the case of quadratic nonlinearity, as in [6], we can write

$$\varepsilon(E) = \varepsilon_0(1 + KE^2) = \varepsilon_0\left(1 + \frac{\mu}{\varepsilon_0}E_0^2\right) = \varepsilon_0 + \mu\frac{|\mathbf{I}|^2}{\varepsilon_0^2} = \varepsilon_0\left(1 + \frac{\mu}{\varepsilon_0^3}|\mathbf{I}|^2\right) = \varepsilon_0\left(1 + k|\mathbf{I}|^2\right),$$

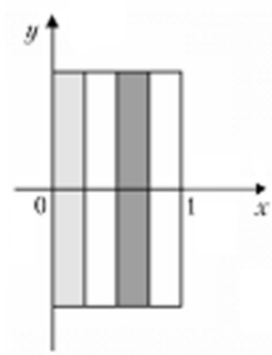

Fig. 2. Laminated material.

where $K = \mu/\varepsilon_0$ and $k = \mu/\varepsilon_0^3$. The values of these quantities are different. The definition of controllability (1) is given in terms of the electric field intensity.

The controllability K_{mix} is calculated by formula (24), which can be written in the form

$$K_{mix} = \lambda \frac{D_1(U)}{D_0} = \lambda U^2 \frac{\langle \mu(x) \, |\nabla N|^4 \rangle}{\langle \varepsilon_0(x) \, |\nabla N|^2 \rangle}. \tag{32}$$

If we assume the period L equal to unity, then $\langle \ \rangle = \int_0^1 dx$.

The cell problem can be solved in a different way. If the layers are perpendicular to the axis $0x$ (Fig. 2), then the function $N(x, y)$ depends on the variable x alone and

$$\frac{\partial N}{\partial x}(x) = \frac{1}{\varepsilon_0(x) \left\langle \frac{1}{\varepsilon_0(x)} \right\rangle}. \tag{33}$$

Substituting (33) into (32), we have

$$K_{mix} = \lambda U^2 \frac{\int_0^1 \mu(x) \frac{1}{\left[\varepsilon_0(x) \left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^4} dx}{\int_0^1 \varepsilon(x) \frac{1}{\left[\varepsilon_0(x) \left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^2} dx} = \lambda U^2 \frac{\int_0^1 \mu(x) \frac{1}{\left[\varepsilon_0(x) \left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^4} dx}{\int_0^1 \frac{1}{\varepsilon_0(x) \left\langle \frac{1}{\varepsilon_0(x)} \right\rangle^2} dx}. \tag{34}$$

The integral in the denominator of (34) is

$$\int_0^1 \frac{1}{\varepsilon_0(x) \left\langle \frac{1}{\varepsilon_0(x)} \right\rangle^2} dx = \frac{1}{\left\langle \frac{1}{\varepsilon_0(x)} \right\rangle^2} \int_0^1 \frac{1}{\varepsilon_0(x)} dx = \frac{1}{\left\langle \frac{1}{\varepsilon_0(x)} \right\rangle}.$$

The last quantity is equal to the averaged dielectric constant across the layer. The integral in the enumerator is

$$\int_0^1 \mu(x) \frac{1}{\left[\varepsilon_0(x) \left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^4} dx = \frac{1}{\left[\left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^4} \int_0^1 \frac{\mu(x)}{\varepsilon_0(x)^4} dx .$$

Then the fraction in (34) is equal to $\left[\left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^{-3} \int_0^1 \frac{\mu(x)}{\varepsilon_0(x)^4} dx$. Grouping the terms in (34), we obtain

$$K_{\text{mix}} = \lambda U^2 \frac{1}{\left[\left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^3} \int_0^1 \frac{\mu(x)}{\varepsilon_0(x)^4} dx = \lambda U^2 \frac{1}{\left[\left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^3} \int_0^1 \frac{K(x)}{\varepsilon_0(x)^3} dx = \lambda U^2 \frac{1}{\left[\left\langle \frac{1}{\varepsilon_0(x)} \right\rangle\right]^3} \left\langle \frac{K(x)}{\varepsilon_0(x)^3} \right\rangle ,$$

where $K(x) = \mu(x)/\varepsilon_0(x)$ denotes the local controllability values (controllability of the composite components).

Denote $\alpha(x) = 1/\varepsilon_0(x)$. In this case, the quadratic term K_{mix} is expressed in terms of $\alpha(x)$ and $K(x)$ in the form

$$K_{\text{mix}} = \lambda U^2 \left[\frac{1}{\langle \alpha(x) \rangle}\right]^3 \langle K(x) (\alpha(x))^3 \rangle = \lambda U^2 \frac{\langle K(x) (\alpha(x))^3 \rangle}{\langle \alpha(x) \rangle^3} . \tag{35}$$

In terms of induction $|\mathbf{I}| = U \frac{1}{\langle \alpha(x) \rangle}$, we write formula (35) as

$$k_{\text{mix}} = \lambda U^2 \left[\frac{1}{\langle \alpha(x) \rangle}\right]^3 \langle K(x) (\alpha(x))^3 \rangle = \lambda \, |\mathbf{I}|^2 \frac{\langle K(x) (\alpha(x))^3 \rangle}{\langle \alpha(x) \rangle} . \tag{36}$$

The local controllability in terms of induction is $K = \mu/\varepsilon_0^3 = k/\varepsilon_0^2 = k\alpha^2$ and (36) takes the form

$$k_{\text{mix}} = \lambda \, |\mathbf{I}|^2 \frac{\langle k(x) (\alpha(x) \rangle}{\langle \alpha(x) \rangle} . \tag{37}$$

From (37) it follows that

$$\min \{k_i\} \le k_{\text{mix}} \le \max \{k_i\} . \tag{38}$$

There are no similar simple estimates for the quantity K_{mix}. For example, the quantity K_{mix} can be larger than the controllability of the components (see the example below).

Two-component composite. Consider a two-component composite, for which

$$\varepsilon_0(x) = \varepsilon_1 , \;\; \mu(x) = \mu_1 \;\; \text{in the material 1} , \;\; \varepsilon_0(x) = \varepsilon_2 , \;\; \mu(x) = \mu_2 \;\; \text{in the material 2} ,$$

and denote by λ_i the specific content of the ith material. For this case,

$$\langle F(x) \rangle = F_1 \lambda_1 + F_2 \lambda_2 = F_1 X + F_2 (1 - X) ,$$

where $X = \lambda_i \in [0, 1]$.

Figure 3 shows the plots of the functions of K_{mix} from (35) and k_{mix} from (37) depending on the volume content of composites (plotted on the horizontal axis is the volume content of material 1). As is seen from Fig. 3a, the controllability of the composite K_{mix} in (35) as a function of the electric field intensity (in terms of which the definition of (1) is given) can be both greater and smaller than the controllability of components K_1 and K_2. The composite controllability as a function of induction k_{mix} in (37) is within the limits of (38). Estimate (38) corresponds to the calculations of the averaged characteristics of the nonlinear laminated composite from [1] (i.e., the estimates from

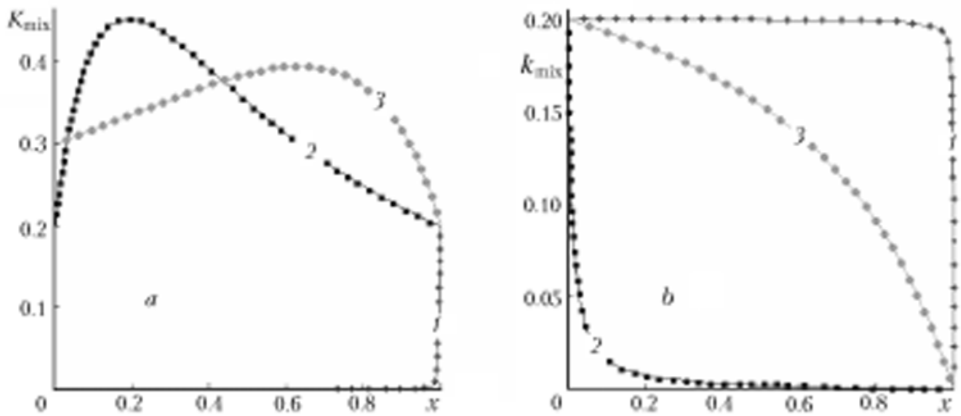

Fig. 3. Controllability K_{mix} (35) as a function of the electric field intensity (a) and controllability k_{mix} (37) as a function of the induction vector (b) depending on the content of components: 1) $\varepsilon_1 = 0.01$, $\varepsilon_2 = 1$, $K_1 = 0.2$, $K_2 = 0$; 2) $\varepsilon_1 = 3$, $\varepsilon_2 = 1$, $K_1 = 0.2$, $K_2 = 0.2$; 3) $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, $K_1 = 0.2$, $K_2 = 0.3$.

[1] correspond to k_{mix} but not to K_{mix}). The quantities $M_{mix} = K_{mix}/\max(K_1, K_2)$ and $m_{mix} = k_{mix}/\max(k_1, k_2)$ are called the controllability strengthening coefficients of composites. They characterize the increase or decrease in the composite controllability of the composite compared to the maximal controllability of its components [9]. As is seen, for the laminated composite M_{mix} can be greater than unity, and m_{mix} never exceeds unity.

Conclusions. A method for calculating the controllability of the "ferroelectric–dielectric" composite by the averaging theory method has been described. For the cases of weak nonlinearities and weak fields, estimates for the averaged controllability of the composite have been obtained and the results of numerical calculations for a two-dimensional model of the composite and exact formulas for calculating the averaged controllability of laminated composites are presented. The results presented explain, in particular, the slow decrease in the averaged controllability of ferroelectrics filled with dielectric particles and predict the possibility of a significant (more than twice) increase in the averaged controllability compared to the controllability of components in laminated "ferroelectric–ferroelectric" composites.

NOTATION

a, A, b, B, constants; $D(E)$, averaged dielectric constant of the composite; D_{total}, flux through the side Γ; $E = |\nabla\varphi|$, electric field intensity modulus; h, discretization step; $\mathbf{I}$, induction vector; $|\mathbf{I}|$, induction vector modulus; I, inclusion (domain occupied by an inclusion); I_2 and I_4, integral functionals; K and k, controllability coefficients of the pure ferroelectric in terms of electric field intensity and in terms of induction, respectively; K_{mix} and k_{mix}, averaged controllability coefficients of the composite in terms of electric field intensity and in terms of induction, respectively; L, M, lengths of the periodicity cell sides; M_{mix}, m_{mix}, controllability amplification coefficients of the composite; $\mathbf{n}$, normal vector; $N(\mathbf{x})$, solution of the cell problem; T, matrix (domain occupied by the matrix); q, volume fraction of the dielectric inclusion; U, potential difference on the periodicity cell faces; $\mathbf{x} = (x, y)$, spatial variable; Y, periodicity cell of the composite; $|Y|$, measure (volume, area) of the periodicity cell Y; $\Gamma = \{-L/2 < x < L/2,\ y = M/2\}$, upper face of the periodicity cell; $\varepsilon(E)$, dielectric constant value corresponding to the value of the electric field intensity E; ε_0, dielectric constant of the ferroelectric at $E \to 0$; ε_{diel}, dielectric constant; $\varphi(\mathbf{x})$, electric field potential; φ_0, zero-order term (term at λ^0) in the expansion for φ; φ_1, first-order term (term at λ); λ, series expansion parameter. Subscripts: mix, mixture; diel, dielectric; total, total.

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